

A. Leray - Hirsch.

Thm. $F \xrightarrow{i} E \xrightarrow{p} B$ fiber bundle,
 R commutative ring. IF

(a) $H^n(F, R)$ is a finite free R -module and

(b) there are classes

$$\boxed{c_j} \in H^{n_j}(E, R)$$

s.t. $\{i^*c_j\}$ forms an R -basis for $H^*(F, R)$,

then

$$H^*(B, R) \otimes_R H^*(F, R) \rightarrow H^*(E, R)$$

$b_j \in H^*(B, R)$

$$\sum b_j \otimes i^*c_j \mapsto \sum p^*b_j \cup c_j$$

is an isomorphism of graded R -mods.

$$(b_j \otimes i^*c_j) \cup (b_l \otimes i^*c_l) = (b_j \cup b_l) \otimes (i^*(c_j \cup c_l))$$

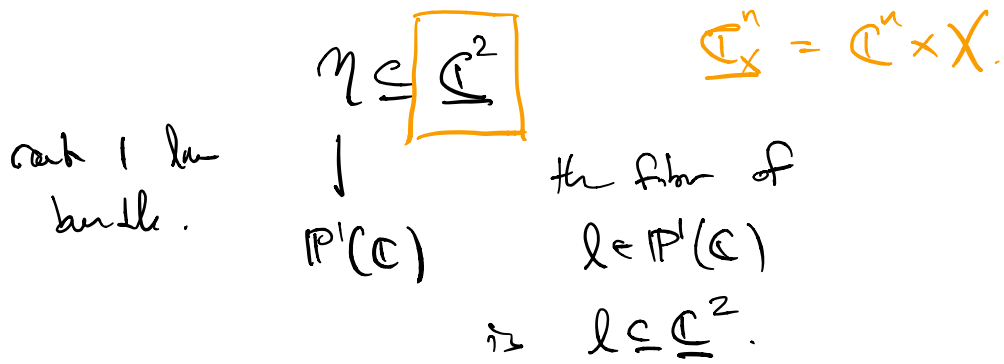
Warning: not generally a map!!! m

B. The Hopf map.

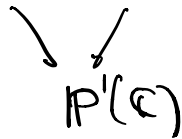
$S^2 \cong \mathbb{P}^1(\mathbb{C}) \stackrel{\mathbb{C}}{=} \hat{=} \text{lines through } \{0\} \text{ in } \mathbb{C}^2.$

$\nearrow = \text{Gr}_1(\mathbb{C}^2)$

Canonical complex line bundle



Look at $X \subseteq \eta$



Our $\ell \in \mathbb{P}^1(\mathbb{C}),$

$X_\ell = \text{unit circle in } \ell \cong \mathbb{C}.$

$$S^1 \longrightarrow X \longrightarrow \mathbb{P}^1(\mathbb{C}) \cong S^2$$

Thm/Fact. $X \cong S^3$.

$$\rightsquigarrow S^1 \longrightarrow S^3 \xrightarrow[\text{map } \eta]{\text{Hopf}} S^2$$

Thm/Fact. n quanta $\rightarrow \pi_3 S^2 \cong \mathbb{Z}$.

Back to Leray-Hirsch.
this is a nonexample.

$$H^*(S^3, \mathbb{Z}) \not\cong H^*(S^2, \mathbb{Z}) \oplus H^*(S^1, \mathbb{Z})$$

$\begin{array}{ccc} & & \underbrace{\hspace{10em}} \\ & & \text{rank } 4 \\ & 2 & \underbrace{\hspace{10em}} \\ & & \underbrace{\hspace{10em}} \\ & & \underbrace{\hspace{10em}} \\ & & \underbrace{\hspace{10em}} \end{array}$

There is no class on S^3 that restricts

to a generator of $H^1(S^1, \mathbb{Z})$

$$\text{since } H^1(S^3, \mathbb{Z}) = 0.$$

C. Flags, Frames, and Grassmannians.

$Gr_k(\mathbb{R}^n) = k\text{-planes in } \mathbb{R}^n.$
 $V \subseteq \mathbb{R}^n$
 sub- \mathbb{R} -vector space.

$Gr_k(\mathbb{C}^n)$

$Gr_k(\mathbb{H}^n)$

Grassmannians.

Compact manifolds.

$Gr_k(\mathbb{R}^\infty)$

\parallel

colim $Gr_k(\mathbb{R}^n)$
 $n \rightarrow \infty$

$Gr_k(\mathbb{R}^n) \rightarrow Gr_k(\mathbb{R}^{n+1})$

$V \subseteq \mathbb{R}^n \mapsto V \cup \{0\} \subseteq \mathbb{R}^{n+1}$

$Gr_k(\mathbb{C}^\infty)$

$Gr_k(\mathbb{H}^\infty)$

Ex. $Gr_1(\mathbb{C}^n) \underset{\text{SII}}{=} \mathbb{P}^{n-1}(\mathbb{C}) = \mathbb{C}P^{n-1}.$

$Gr_{n-1}(\mathbb{C}^n)$

$V_k(\mathbb{R}^n) =$ orthogonal k -frames in \mathbb{R}^n

(v_1, \dots, v_k) s.t.

$$\langle v_i, v_j \rangle = \delta_{ij}.$$

$V_k(\mathbb{C}^n)$

$V_k(\mathbb{H}^n)$

Stiefel
varieties

Stiefel manifolds

$$V_1(\mathbb{C}^n) \cong S^{2n-1}$$

$F_k(\mathbb{R}^n) =$ flags of subspaces

$$0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_k \subseteq \mathbb{R}^n$$

$$\dim V_i = i.$$

$F_k(\mathbb{C}^n)$

$F_k(\mathbb{H}^n)$

$$V_1(\mathbb{C}^n) = Gr_1(\mathbb{C}^n)$$

S^1

$$\mathbb{P}^{n-1}(\mathbb{C})$$

D. Lots of fibrations.

I.

$$\begin{array}{ccc}
 U(k) \longrightarrow V_k(\mathbb{R}^n) & \xrightarrow{\text{span}} & Gr_k(\mathbb{R}^n) \\
 \{v_1, \dots, v_k\} & \longleftarrow & \text{subspace spanned} \\
 \langle v_i, v_j \rangle = \delta_{ij} & & \text{by } \{v_i\}
 \end{array}$$

$$\{e_1, \dots, e_k\} \subseteq \mathbb{R}^n$$

$$U(k) \longrightarrow V_k(\mathbb{C}^n) \longrightarrow Gr_k(\mathbb{C}^n)$$

$$Sp(2k) \longrightarrow V_k(\mathbb{H}^n) \longrightarrow Gr_k(\mathbb{H}^n)$$

Ia (k=1, n=∞)

$$\begin{array}{ccc}
 U(1) \longrightarrow V_1(\mathbb{R}^\infty) \longrightarrow Gr_1(\mathbb{R}^\infty) \\
 \mathbb{S}^1 & \mathbb{S}^1 & \mathbb{S}^1 \\
 \mathbb{Z}/2 & \mathbb{S}^\infty & \mathbb{RP}^\infty \\
 & \mathbb{S}^1 & \mathbb{S}^1 \\
 & * & \mathbb{B}\mathbb{Z}/2
 \end{array}$$

$$(\mathbb{Q} \mathbb{RP}^\infty \simeq \mathbb{Z}/2)$$

$$\begin{array}{ccc}
 U(1) \longrightarrow V_1(\mathbb{C}^\infty) \longrightarrow Gr_1(\mathbb{C}^\infty) \\
 \mathbb{S}^1 & \mathbb{S}^1 & \mathbb{S}^1 \\
 \mathbb{S}^1 & \mathbb{S}^\infty & \mathbb{C}\mathbb{P}^\infty \\
 & \mathbb{S}^1 & \mathbb{S}^1 \\
 & * & \mathbb{B}\mathbb{S}^1
 \end{array}$$

$$\begin{array}{ccccc}
 Sp(2) & \longrightarrow & V_1(\mathbb{H}^\infty) & \longrightarrow & Gr_1(\mathbb{H}^\infty) \\
 \downarrow \text{SI} & & \downarrow \text{SI} & & \downarrow \text{SI} \\
 S^3 & & \mathbb{R} & & \mathbb{H}P^\infty \\
 & & & & \downarrow \text{SI} \\
 & & & & BS^3
 \end{array}$$

IIb. In general,

$$\begin{array}{ccc}
 V_k(\mathbb{R}^\infty) & \Rightarrow & \Omega Gr_k(\mathbb{R}^\infty) \\
 \downarrow \text{SI} & & \downarrow \text{SI} \\
 \vdots & & O(k).
 \end{array}$$

$$\text{II. } V_{j-k}(\mathbb{R}^{n-k}) \rightarrow V_j(\mathbb{R}^n) \xrightarrow[k \leq j]{\text{First } k} \underline{V_k(\mathbb{R}^n)}$$

$$V_{j-k}(\mathbb{C}^{n-k}) \rightarrow V_j(\mathbb{C}^n) \rightarrow V_k(\mathbb{C}^n)$$

$$V_{j-k}(\mathbb{H}^{n-k}) \rightarrow V_j(\mathbb{H}^n) \rightarrow V_k(\mathbb{H}^n)$$

$$\text{IIa (k=1)} \quad \underline{V_{j-1}(\mathbb{R}^{n-1})} \rightarrow \underline{V_j(\mathbb{R}^n)} \rightarrow S^{n-1}$$

$$V_{j-1}(\mathbb{C}^{n-1}) \rightarrow V_j(\mathbb{C}^n) \rightarrow S^{2n-1}$$

$$V_{j-1}(\mathbb{H}^{n-1}) \rightarrow V_j(\mathbb{H}^n) \rightarrow S^{4n-1}$$

$$\text{II b } (j=n) \quad V_{n-k}(\mathbb{R}^{n+k}) \xrightarrow{\quad} V_n(\mathbb{R}^n) \xrightarrow{\quad} V_k(\mathbb{R}^n)$$

$$\begin{array}{ccc} \text{SII} & \text{SII} & \text{II} \\ \hline O(n-k) \times O(k) & \xrightarrow{\quad} & V_k(\mathbb{R}^n) \end{array}$$

$$\text{II c } (j=n, k=1) \quad O(n-1) \xrightarrow{\quad} O(n) \xrightarrow{\quad} S^{n-1}$$

$$U(n-1) \xrightarrow{\quad} U(n) \xrightarrow{\quad} S^{2n-1}$$

$$Sp(2n-2) \xrightarrow{\quad} Sp(2n) \xrightarrow{\quad} S^{4n-1}$$

$$\text{III} \quad F_k(\mathbb{R}^k) \xrightarrow{\quad} F_k(\mathbb{R}^n) \xrightarrow{\quad} Gr_k(\mathbb{R}^n)$$

$$\begin{array}{c} ; \\ ; \end{array} \quad (0 \oplus V_1 \oplus \dots \oplus V_k) \xrightarrow{\quad} V_k$$

$$\text{IV} \quad \frac{O(n)}{O(n-k) \times O(k)} \cong Gr_k(\mathbb{R}^n)$$

$$\frac{U(n)}{U(n-k) \times U(k)} \cong Gr_k(\mathbb{C}^n)$$

A	0
0	B

$$\frac{Sp(2n)}{Sp(2n-2k) \times Sp(2k)} \cong Gr_k(\mathbb{H}^n).$$