# A. Leray–Hirsch.

**Thm.** $F \xrightarrow{i} E \xrightarrow{p} B$ fiber bundle, $R$ commutative ring. If

(a) $H^n(F,R)$ is a finite free $R$-module and

(b) there are classes
$$\boxed{c_j} \in H^{n_j}(E,R)$$
s.t. $\{i^*c_j\}$ forms an $R$-basis for $H^*(F,R)$,

then
$$H^*(B,R) \otimes_R H^*(F,R) \to H^*(E,R)$$
$b_j \in H^*(B,R)$
$$\sum b_j \otimes i^* c_j \longmapsto \sum p^* b_j \cup c_j$$
is an isomorphism of graded $R$-mods.

$(b_j \otimes i^* c_j) \cup (b_\ell \otimes i^* c_\ell) = (b_j \cup b_\ell) \otimes (i^*(c_j \cup c_\ell))$

Warning: not generally a vb map!!!

## B. The Hopf map.

$$S^2 \cong \mathbb{P}^1(\mathbb{C}) = \mathbb{C}\text{-lines through } \{0\} \text{ in } \mathbb{C}^2.$$

$$= Gr_1(\mathbb{C}^2)$$

Canonical complex line bundle

$$\underline{\mathbb{C}}^n_X = \mathbb{C}^n \times X.$$

$$\begin{array}{c} \eta \subseteq \boxed{\underline{\mathbb{C}}^2} \\ \downarrow \\ \mathbb{P}^1(\mathbb{C}) \end{array}$$

rank 1 line bundle.

the fiber of $\ell \in \mathbb{P}^1(\mathbb{C})$ is $\ell \subseteq \mathbb{C}^2$.

Look at
$$\begin{tikzcd} X \arrow[dr] & \subseteq & \eta \arrow[dl] \\ & \mathbb{P}^1(\mathbb{C}) & \end{tikzcd}$$

Over $\ell \in \mathbb{P}^1(\mathbb{C})$,

$$X_\ell = \text{unit circle in } \ell \cong \mathbb{C}.$$

$$S^1 \longrightarrow X \longrightarrow \mathbb{P}^1(\mathbb{C}) \cong S^2$$

Thm/Fact. $X \cong S^3$.

$$\rightsquigarrow \quad S^1 \longrightarrow S^3 \xrightarrow[\text{map } \eta]{\text{Hopf}} S^2.$$

Thm/Fact. $\eta$ generates $\pi_3 S^2 \cong \mathbb{Z}$.

Back to Leray–Hirsch.

This is a nonexample.

$$\underset{2}{H^*(S^3,\mathbb{Z})} \not\cong \underbrace{\underset{2}{H^*(S^2,\mathbb{Z})} \otimes_{\mathbb{Z}} \underset{2}{H^*(S^1,\mathbb{Z})}}_{\text{rank } 4}$$

There is no class on $S^3$ that restricts to a generator of $H^1(S^1,\mathbb{Z})$ since $H^1(S^3,\mathbb{Z})=0$.

# C. Flags, Frames, and Grassmannians.

$Gr_k(\mathbb{R}^n)$ = $k$-planes in $\mathbb{R}^n$.

$V \subseteq \mathbb{R}^n$

sub-$\mathbb{R}$-vector space.

$Gr_k(\mathbb{C}^n)$

$Gr_k(\mathbb{H}^n)$

Grassmannians.

Compact manifolds.

$$Gr_k(\mathbb{R}^\infty) = \operatorname*{colim}_{n\to\infty} Gr_k(\mathbb{R}^n)$$

$$Gr_k(\mathbb{R}^n) \longrightarrow Gr_k(\mathbb{R}^{n+1})$$

$$V \subseteq \mathbb{R}^n \longmapsto V \oplus 0 \subseteq \mathbb{R}^n \oplus \mathbb{R}$$

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$Gr_k(\mathbb{C}^\infty)$

$Gr_k(\mathbb{H}^\infty)$

**Ex.** $Gr_1(\mathbb{C}^n) = \mathbb{P}^{n-1}(\mathbb{C}) = \mathbb{C}P^{n-1}$.

$$Gr_1(\mathbb{C}^n) \cong Gr_{n-1}(\mathbb{C}^n)$$

$V_k(\mathbb{R}^n)$ = orthogonal $k$-frames in $\mathbb{R}^n$

$(v_1, \ldots, v_k)$ s.t.

$\langle v_i, v_j \rangle = \delta_{ij}$.

$V_k(\mathbb{C}^n)$

$V_1(\mathbb{C}^n) \cong S^{2n-1}$

$V_k(\mathbb{H}^n)$

Stiefel ~~varieties~~

Stiefel manifolds

$F_k(\mathbb{R}^n)$ = flags of subspaces

$0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k \subseteq \mathbb{R}^n$

$\dim V_i = i$.

$F_k(\mathbb{C}^n)$

$F_k(\mathbb{H}^n)$

$$V_1(\mathbb{C}^n) = Gr_1(\mathbb{C}^n) \cong \mathbb{P}^{n-1}(\mathbb{C})$$

# D. Lots of Fibrations.

I.

$$O(k) \to V_k(\mathbb{R}^n) \xrightarrow{\text{span}} Gr_k(\mathbb{R}^n)$$

$\{v_1, \dots, v_k\} \longmapsto$ subspace spanned by $\{v_i\}$

$\langle v_i, v_j \rangle = \delta_{ij}$

$\{e_1, \dots, e_k\} \subseteq \mathbb{R}^n$

$$U(k) \to V_k(\mathbb{C}^n) \longrightarrow Gr_k(\mathbb{C}^n)$$

$$Sp(2k) \to V_k(\mathbb{H}^n) \longrightarrow Gr_k(\mathbb{H}^n)$$

Ia $(k=1, n=\infty)$

$$\begin{array}{ccccc} O(1) & \to & V_1(\mathbb{R}^\infty) & \to & Gr_1(\mathbb{R}^\infty) \\ \cong & & \cong & & \cong \\ \mathbb{Z}/2 & & S^\infty & & \mathbb{RP}^\infty \\ & & \simeq & & \cong \\ & & * & & B\mathbb{Z}/2 \end{array}$$

$(\Omega \mathbb{RP}^\infty \simeq \mathbb{Z}/2)$

$$\begin{array}{ccccc} U(1) & \to & V_1(\mathbb{C}^\infty) & \to & Gr_1(\mathbb{C}^\infty) \\ \cong & & \cong & & \cong \\ S^1 & & S^\infty & & \mathbb{CP}^\infty \\ & & \simeq & & \cong \\ & & * & & BS^1 \end{array}$$

$$
\begin{array}{ccccc}
Sp(2) & \longrightarrow & V_1(\mathbb{H}^\infty) & \longrightarrow & Gr_1(\mathbb{H}^\infty) \\
\wr\!\wr & & \wr\!\wr & & \wr\!\wr \\
S^3 & & * & & \mathbb{HP}^\infty \\
 & & & & \wr\!\wr \\
 & & & & BS^3
\end{array}
$$

Ib. In general,

$$
\begin{array}{ccc}
V_k(\mathbb{R}^\infty) & \Longrightarrow & \Omega\, Gr_k(\mathbb{R}^\infty) \\
\wr\!\wr & & \wr\!\wr \\
* & & O(k).
\end{array}
$$

$\vdots$

II. $V_{j-k}(\mathbb{R}^{n-k}) \to V_j(\mathbb{R}^n) \xrightarrow[k \leq j]{\text{first } k} V_{\underline{\underline{k}}}(\mathbb{R}^n)$

$$V_{j-k}(\mathbb{C}^{n-k}) \to V_j(\mathbb{C}^n) \longrightarrow V_k(\mathbb{C}^n)$$

$$V_{j-k}(\mathbb{H}^{n-k}) \quad V_j(\mathbb{H}^n) \longrightarrow V_k(\mathbb{H}^n)$$

IIa $(k=1)$ $\quad V_{j-1}(\mathbb{R}^{n-1}) \longrightarrow V_j(\mathbb{R}^n) \longrightarrow S^{n-1}$

$$V_{j-1}(\mathbb{C}^{n-1}) \longrightarrow V_j(\mathbb{C}^n) \longrightarrow S^{2n-1}$$

$$V_{j-1}(\mathbb{H}^{n-1}) \longrightarrow V_j(\mathbb{H}^n) \longrightarrow S^{4n-1}$$

II b $(j=n)$

$$\begin{tikzcd} V_{n-k}(\mathbb{R}^{n-k}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & V_n(\mathbb{R}^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & V_k(\mathbb{R}^n) \arrow[d, phantom, "="] \\ O(n-k) \arrow[r] & O(n) \arrow[r] & V_k(\mathbb{R}^n) \end{tikzcd}$$

II c $(j=n, k=1)$

$$O(n-1) \to O(n) \to S^{n-1}$$

$$U(n-1) \to U(n) \to S^{2n-1}$$

$$Sp(2n-2) \to Sp(2n) \to S^{4n-1}.$$

III.

$$F_k(\mathbb{R}^k) \to F_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n).$$

$$(0 \subseteq V_1 \subseteq \cdots \subseteq V_k) \longmapsto V_k$$

IV.

$$\frac{O(n)}{O(n-k) \times O(k)} \cong Gr_k(\mathbb{R}^n)$$

$$\frac{U(n)}{U(n-k) \times U(k)} \cong Gr_k(\mathbb{C}^n)$$

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$$\frac{Sp(2n)}{Sp(2n-2k)\times Sp(2k)} \cong Gr_k(\mathbb{H}^n).$$